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Three-quark exchange operators, crossing matrices and Fierz transformations in SU(2) and SU(3)

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Abstract

We give explicit expressions for the three-quark exchange operators, crossing matrices and Fierz transforms for the SU(2) and SU(3) groups. We identify the invariant terms in these operators and express them in terms of Casimir operators.

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I. INTRODUCTION

Dirac [1] was the first to express the two-particle spin-exchange operator

$$\mathrm{P}_{12} = \frac{1}{2} + \frac{1}{2}\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \tag{1}$$

in terms of Pauli matrices $\boldsymbol{\tau}_{1,2}$ of the two particles, where $\boldsymbol{\tau} \cdot \boldsymbol{\tau} = \sum_{a=1}^{3} \tau^a \tau^a$. This, and the following result

$$\mathrm{P}_{12}\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 = \frac{3}{2} - \frac{1}{2}\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \tag{2}$$

are equivalent to a "Fierz reordering formula" for the quartic field interaction, or to the SU(2) crossing matrix

$$\mathbf{C} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} . \tag{3}$$

for two-body processes. The same trick has been extended to the SU(3) Lie algebra

$$\mathrm{P}_{12} = \frac{1}{3} + \frac{1}{2}\boldsymbol{\lambda}_1 \cdot \boldsymbol{\lambda}_2 \tag{4a}$$

$$\mathrm{P}_{12}\boldsymbol{\lambda}_1 \cdot \boldsymbol{\lambda}_2 = \frac{16}{9} - \frac{1}{3}\boldsymbol{\lambda}_1 \cdot \boldsymbol{\lambda}_2 \tag{4b}$$

again only for two particles [2], with the resulting crossing matrix (or the equivalent Fierz reordering formulas for bilinear products of Gell-Mann matrices) being

$$\mathbf{C} = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{16}{9} & -\frac{1}{3} \end{pmatrix} . \tag{5}$$

Here the lower index indicates the number of the quark, λ^a are the Gell-Mann matrices, and f^{abc}, d^{abc} are the usual SU(3) structure constants.

In the meantime a need has arisen for tri-linear Fierz formulas/crossing relations in connection with applications of the three-flavour 't Hooft interaction [3]. Such relations don't seem to be extant in the literature [4–6]. In this note we present the corresponding three-body exchange operators for quarks (particles in the fundamental representation of SU(2) and/or SU(3)), as well as the equivalent Fierz reordering formulas for the sextic field interaction.

II. THREE-QUARK EXCHANGE OPERATORS

A. The SU(2) algebra

1. Three-body exchange operators

Using the symmetric group $\mathcal{S}_3$ it is straightforward, if tedious, to derive the SU(2) version of the three-quark/spin exchange operator

$$\begin{aligned} \mathrm{P}_{123} &= \mathrm{P}_{23}\mathrm{P}_{12} \\ &= \frac{1}{4} + \frac{1}{4}\sum_{i<j}^{3} \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j + \frac{i}{4}\varepsilon^{abc}\boldsymbol{\tau}_1^a\boldsymbol{\tau}_2^b\boldsymbol{\tau}_3^c \end{aligned} \tag{6a}$$

$$\begin{aligned} \mathrm{P}_{132} &= \mathrm{P}_{123}^2 \\ &= \frac{1}{4} + \frac{1}{4}\sum_{i<j}^{3} \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j - \frac{i}{4}\varepsilon^{abc}\boldsymbol{\tau}_1^a\boldsymbol{\tau}_2^b\boldsymbol{\tau}_3^c \end{aligned} \tag{6b}$$

Similar results are

$$\mathrm{P}_{123}\sum_{i<j}^{3} \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j = \frac{1}{2}\left(9 + \sum_{i<j}^{3} \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j - 3i\varepsilon^{abc}\boldsymbol{\tau}_1^a\boldsymbol{\tau}_2^b\boldsymbol{\tau}_3^c\right) \tag{7a}$$

$$\mathrm{P}_{132}\sum_{i<j}^{3} \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j = \frac{1}{4}\left(9 + \sum_{i<j}^{3} \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j + 3i\varepsilon^{abc}\boldsymbol{\tau}_1^a\boldsymbol{\tau}_2^b\boldsymbol{\tau}_3^c\right) \tag{7b}$$

as well as

$$i\mathrm{P}_{123}\varepsilon^{abc}\boldsymbol{\tau}_1^a\boldsymbol{\tau}_2^b\boldsymbol{\tau}_3^c = \frac{1}{2}\left(-3 + \sum_{i<j}^{3} \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j - i\varepsilon^{abc}\boldsymbol{\tau}_1^a\boldsymbol{\tau}_2^b\boldsymbol{\tau}_3^c\right) \tag{8a}$$

$$i\mathrm{P}_{132}\varepsilon^{abc}\boldsymbol{\tau}_1^a\boldsymbol{\tau}_2^b\boldsymbol{\tau}_3^c = \frac{1}{2}\left(3 - \sum_{i<j}^{3} \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j - i\varepsilon^{abc}\boldsymbol{\tau}_1^a\boldsymbol{\tau}_2^b\boldsymbol{\tau}_3^c\right) \tag{8b}$$

2. Crossing matrix

These results are summarized by the crossing matrices

$$\mathbf{C} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 \\ 9 & 1 & -3 \\ -6 & 2 & -2 \end{pmatrix}, \tag{9}$$

for P_{123}, and

$$\mathbf{C}^2 = \frac{1}{4}\begin{pmatrix} 1 & 1 & -1 \\ 9 & 1 & 3 \\ 6 & -2 & -2 \end{pmatrix}, \tag{10}$$

for P_{132}. A valuable check is the constraint $\mathbf{C}^3 = \mathit{1}$.

3. Three-body Fierz identities

$$\begin{aligned}\delta_{\alpha\delta}\delta_{\gamma\rho}\delta_{\sigma\beta} = \frac{1}{4}\Big(&\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{\sigma\rho} + \delta_{\alpha\beta}\boldsymbol{\tau}_{\gamma\delta}\cdot\boldsymbol{\tau}_{\sigma\rho} + \delta_{\gamma\delta}\boldsymbol{\tau}_{\alpha\beta}\cdot\boldsymbol{\tau}_{\sigma\rho} \\ &+ \delta_{\sigma\rho}\boldsymbol{\tau}_{\gamma\delta}\cdot\boldsymbol{\tau}_{\alpha\beta} + i\varepsilon^{abc}\boldsymbol{\tau}^a_{\alpha\beta}\boldsymbol{\tau}^b_{\gamma\delta}\boldsymbol{\tau}^c_{\sigma\rho}\Big)\end{aligned} \tag{11a}$$

$$\begin{aligned}\delta_{\alpha\rho}\delta_{\gamma\beta}\delta_{\sigma\delta} = \frac{1}{4}\Big(&\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{\sigma\rho} + \delta_{\alpha\beta}\boldsymbol{\tau}_{\gamma\delta}\cdot\boldsymbol{\tau}_{\sigma\rho} + \delta_{\gamma\delta}\boldsymbol{\tau}_{\alpha\beta}\cdot\boldsymbol{\tau}_{\sigma\rho} \\ &+ \delta_{\sigma\rho}\boldsymbol{\tau}_{\gamma\delta}\cdot\boldsymbol{\tau}_{\alpha\beta} - i\varepsilon^{abc}\boldsymbol{\tau}^a_{\alpha\beta}\boldsymbol{\tau}^b_{\gamma\delta}\boldsymbol{\tau}^c_{\sigma\rho}\Big)\end{aligned} \tag{11b}$$

$$\begin{aligned}\delta_{\alpha\delta}\boldsymbol{\tau}_{\gamma\rho}\cdot\boldsymbol{\tau}_{\sigma\beta} + \delta_{\gamma\rho}\boldsymbol{\tau}_{\alpha\delta}\cdot\boldsymbol{\tau}_{\sigma\beta} + \delta_{\sigma\beta}\boldsymbol{\tau}_{\gamma\rho}\cdot\boldsymbol{\tau}_{\alpha\delta} = \frac{1}{4}\Big(&9\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{\sigma\rho} \\ + \delta_{\alpha\beta}\boldsymbol{\tau}_{\gamma\delta}\cdot\boldsymbol{\tau}_{\sigma\rho} + \delta_{\gamma\delta}\boldsymbol{\tau}_{\alpha\beta}\cdot\boldsymbol{\tau}_{\sigma\rho} + \delta_{\sigma\rho}\boldsymbol{\tau}_{\gamma\delta}\cdot\boldsymbol{\tau}_{\alpha\beta} - 3i\varepsilon^{abc}\boldsymbol{\tau}^a_{\alpha\beta}\boldsymbol{\tau}^b_{\gamma\delta}\boldsymbol{\tau}^c_{\sigma\rho}\Big)&\end{aligned} \tag{12a}$$

$$\begin{aligned}\delta_{\alpha\rho}\boldsymbol{\tau}_{\gamma\beta}\cdot\boldsymbol{\tau}_{\sigma\delta} + \delta_{\gamma\beta}\boldsymbol{\tau}_{\alpha\rho}\cdot\boldsymbol{\tau}_{\sigma\delta} + \delta_{\sigma\beta}\boldsymbol{\tau}_{\gamma\rho}\cdot\boldsymbol{\tau}_{\alpha\delta} = \frac{1}{4}\Big(&9\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{\sigma\rho} \\ + \delta_{\alpha\beta}\boldsymbol{\tau}_{\gamma\delta}\cdot\boldsymbol{\tau}_{\sigma\rho} + \delta_{\gamma\delta}\boldsymbol{\tau}_{\alpha\beta}\cdot\boldsymbol{\tau}_{\sigma\rho} + \delta_{\sigma\rho}\boldsymbol{\tau}_{\gamma\delta}\cdot\boldsymbol{\tau}_{\alpha\beta} + 3i\varepsilon^{abc}\boldsymbol{\tau}^a_{\alpha\beta}\boldsymbol{\tau}^b_{\gamma\delta}\boldsymbol{\tau}^c_{\sigma\rho}\Big)&\end{aligned} \tag{12b}$$

$$\begin{aligned}i\varepsilon^{abc}\boldsymbol{\tau}^a_{\alpha\delta}\boldsymbol{\tau}^b_{\gamma\rho}\boldsymbol{\tau}^c_{\sigma\beta} = \frac{1}{2}\Big(&-3\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{\sigma\rho} \\ + \delta_{\alpha\beta}\boldsymbol{\tau}_{\gamma\delta}\cdot\boldsymbol{\tau}_{\sigma\rho} + \delta_{\gamma\delta}\boldsymbol{\tau}_{\alpha\beta}\cdot\boldsymbol{\tau}_{\sigma\rho} + \delta_{\sigma\rho}\boldsymbol{\tau}_{\gamma\delta}\cdot\boldsymbol{\tau}_{\alpha\beta} - i\varepsilon^{abc}\boldsymbol{\tau}^a_{\alpha\beta}\boldsymbol{\tau}^b_{\gamma\delta}\boldsymbol{\tau}^c_{\sigma\rho}\Big)&\end{aligned} \tag{13a}$$

$$\begin{aligned}i\varepsilon^{abc}\boldsymbol{\tau}^a_{\alpha\rho}\boldsymbol{\tau}^b_{\gamma\beta}\boldsymbol{\tau}^c_{\sigma\delta} = \frac{1}{2}\Big(&3\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{\sigma\rho} \\ - (\delta_{\alpha\beta}\boldsymbol{\tau}_{\gamma\delta}\cdot\boldsymbol{\tau}_{\sigma\rho} + \delta_{\gamma\delta}\boldsymbol{\tau}_{\alpha\beta}\cdot\boldsymbol{\tau}_{\sigma\rho} + \delta_{\sigma\rho}\boldsymbol{\tau}_{\gamma\delta}\cdot\boldsymbol{\tau}_{\alpha\beta}) - if^{abc}\boldsymbol{\tau}^a_{\alpha\beta}\boldsymbol{\tau}^b_{\gamma\delta}\boldsymbol{\tau}^c_{\sigma\rho}\Big)&\end{aligned} \tag{13b}$$

B. The SU(3) algebra

1. Three-body exchange operators

Similarly, we have

$$\mathrm{P}_{123} = \frac{1}{9} + \frac{1}{6}\sum_{i<j}^{3} \boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j + \frac{1}{4} d^{abc} \boldsymbol{\lambda}_1^a \boldsymbol{\lambda}_2^b \boldsymbol{\lambda}_3^c + \frac{i}{4} f^{abc} \boldsymbol{\lambda}_1^a \boldsymbol{\lambda}_2^b \boldsymbol{\lambda}_3^c \tag{14a}$$

$$\mathrm{P}_{132} = \frac{1}{9} + \frac{1}{6}\sum_{i<j}^{3} \boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j + \frac{1}{4} d^{abc} \boldsymbol{\lambda}_1^a \boldsymbol{\lambda}_2^b \boldsymbol{\lambda}_3^c - \frac{1}{4} i f^{abc} \boldsymbol{\lambda}_1^a \boldsymbol{\lambda}_2^b \boldsymbol{\lambda}_3^c \tag{14b}$$

as well as similar relations for the operators

$$\mathrm{P}_{123}\sum_{i<j}^{3} \boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j = \frac{1}{9} + \frac{1}{6}\sum_{i<j}^{3} \boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j + \frac{1}{4} d^{abc} \boldsymbol{\lambda}_1^a \boldsymbol{\lambda}_2^b \boldsymbol{\lambda}_3^c + \frac{i}{4} f^{abc} \boldsymbol{\lambda}_1^a \boldsymbol{\lambda}_2^b \boldsymbol{\lambda}_3^c \tag{15a}$$

$$\mathrm{P}_{132}\sum_{i<j}^{3} \boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j = \frac{1}{9} + \frac{1}{6}\sum_{i<j}^{3} \boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j + \frac{1}{4} d^{abc} \boldsymbol{\lambda}_1^a \boldsymbol{\lambda}_2^b \boldsymbol{\lambda}_3^c - \frac{1}{4} i f^{abc} \boldsymbol{\lambda}_1^a \boldsymbol{\lambda}_2^b \boldsymbol{\lambda}_3^c \tag{15b}$$

and

$$\mathrm{P}_{123} d^{abc} \boldsymbol{\lambda}_1^a \boldsymbol{\lambda}_2^b \boldsymbol{\lambda}_3^c = \frac{80}{81} - \frac{5}{27}\sum_{i<j}^{3} \boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j + \frac{13}{18} d^{abc} \boldsymbol{\lambda}_1^a \boldsymbol{\lambda}_2^b \boldsymbol{\lambda}_3^c - \frac{5}{18} i f^{abc} \boldsymbol{\lambda}_1^a \boldsymbol{\lambda}_2^b \boldsymbol{\lambda}_3^c \tag{16a}$$

$$\mathrm{P}_{132} d^{abc} \boldsymbol{\lambda}_1^a \boldsymbol{\lambda}_2^b \boldsymbol{\lambda}_3^c = \frac{80}{81} - \frac{5}{27}\sum_{i<j}^{3} \boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j + \frac{13}{18} d^{abc} \boldsymbol{\lambda}_1^a \boldsymbol{\lambda}_2^b \boldsymbol{\lambda}_3^c + \frac{5}{18} i f^{abc} \boldsymbol{\lambda}_1^a \boldsymbol{\lambda}_2^b \boldsymbol{\lambda}_3^c \tag{16b}$$

$$\mathrm{P}_{123} i f^{abc} \boldsymbol{\lambda}_1^a \boldsymbol{\lambda}_2^b \boldsymbol{\lambda}_3^c = -\frac{16}{9} + \frac{1}{3}\sum_{i<j}^{3} \boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j + \frac{1}{2} d^{abc} \boldsymbol{\lambda}_1^a \boldsymbol{\lambda}_2^b \boldsymbol{\lambda}_3^c - \frac{1}{2} i f^{abc} \boldsymbol{\lambda}_1^a \boldsymbol{\lambda}_2^b \boldsymbol{\lambda}_3^c \tag{16c}$$

$$\mathrm{P}_{132} i f^{abc} \boldsymbol{\lambda}_1^a \boldsymbol{\lambda}_2^b \boldsymbol{\lambda}_3^c = \frac{16}{9} - \frac{1}{3}\sum_{i<j}^{3} \boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j - \frac{1}{2} d^{abc} \boldsymbol{\lambda}_1^a \boldsymbol{\lambda}_2^b \boldsymbol{\lambda}_3^c - \frac{1}{2} i f^{abc} \boldsymbol{\lambda}_1^a \boldsymbol{\lambda}_2^b \boldsymbol{\lambda}_3^c \tag{16d}$$

2. Crossing matrix

This leads to the following first cyclic permutation three-quark crossing matrix

$$\mathbf{C} = \begin{pmatrix} \frac{1}{9} & \frac{1}{6} & \frac{1}{4} & \frac{1}{4} \\ \frac{16}{9} & \frac{2}{3} & -\frac{1}{2} & \frac{1}{2} \\ \frac{80}{81} & -\frac{5}{27} & \frac{13}{18} & -\frac{5}{18} \\ -\frac{16}{9} & \frac{1}{3} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} . \tag{17}$$

Similarly, for the second cyclic permutation we find

$$\mathbf{C}^2 = \begin{pmatrix} \frac{1}{9} & \frac{1}{6} & \frac{1}{4} & -\frac{1}{4} \\ \frac{16}{9} & \frac{2}{3} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{80}{81} & -\frac{5}{27} & \frac{13}{18} & \frac{5}{18} \\ \frac{16}{9} & -\frac{1}{3} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} , \tag{18}$$

and, of course satisfying $\mathbf{C}^3 = \mathit{1}$.

3. Three-body Fierz identities

$$\begin{aligned} \delta_{\alpha\delta}\delta_{\gamma\rho}\delta_{\sigma\beta} = &\frac{1}{9}\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{\sigma\rho} + \frac{1}{6}\left(\delta_{\alpha\beta}\boldsymbol{\lambda}_{\gamma\delta}\cdot\boldsymbol{\lambda}_{\sigma\rho} + \delta_{\gamma\delta}\boldsymbol{\lambda}_{\alpha\beta}\cdot\boldsymbol{\lambda}_{\sigma\rho} + \delta_{\sigma\rho}\boldsymbol{\lambda}_{\gamma\delta}\cdot\boldsymbol{\lambda}_{\alpha\beta}\right) \\ &+ \frac{1}{4}d^{abc}\boldsymbol{\lambda}^a_{\alpha\beta}\boldsymbol{\lambda}^b_{\gamma\delta}\boldsymbol{\lambda}^c_{\sigma\rho} + \frac{1}{4}if^{abc}\boldsymbol{\lambda}^a_{\alpha\beta}\boldsymbol{\lambda}^b_{\gamma\delta}\boldsymbol{\lambda}^c_{\sigma\rho} \end{aligned} \tag{19a}$$

$$\begin{aligned} \delta_{\alpha\rho}\delta_{\gamma\beta}\delta_{\sigma\delta} = &\frac{1}{9}\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{\sigma\rho} + \frac{1}{6}\left(\delta_{\alpha\beta}\boldsymbol{\lambda}_{\gamma\delta}\cdot\boldsymbol{\lambda}_{\sigma\rho} + \delta_{\gamma\delta}\boldsymbol{\lambda}_{\alpha\beta}\cdot\boldsymbol{\lambda}_{\sigma\rho} + \delta_{\sigma\rho}\boldsymbol{\lambda}_{\gamma\delta}\cdot\boldsymbol{\lambda}_{\alpha\beta}\right) \\ &+ \frac{1}{4}d^{abc}\boldsymbol{\lambda}^a_{\alpha\beta}\boldsymbol{\lambda}^b_{\gamma\delta}\boldsymbol{\lambda}^c_{\sigma\rho} - \frac{1}{4}if^{abc}\boldsymbol{\lambda}^a_{\alpha\beta}\boldsymbol{\lambda}^b_{\gamma\delta}\boldsymbol{\lambda}^c_{\sigma\rho} \end{aligned} \tag{19b}$$

$$\begin{aligned} &\delta_{\alpha\delta}\boldsymbol{\lambda}_{\gamma\rho}\cdot\boldsymbol{\lambda}_{\sigma\beta} + \delta_{\gamma\rho}\boldsymbol{\lambda}_{\alpha\delta}\cdot\boldsymbol{\lambda}_{\sigma\beta} + \delta_{\sigma\beta}\boldsymbol{\lambda}_{\gamma\rho}\cdot\boldsymbol{\lambda}_{\alpha\delta} \\ = &\frac{2}{3}\left(\delta_{\alpha\beta}\boldsymbol{\lambda}_{\gamma\delta}\cdot\boldsymbol{\lambda}_{\sigma\rho} + \delta_{\gamma\delta}\boldsymbol{\lambda}_{\alpha\beta}\cdot\boldsymbol{\lambda}_{\sigma\rho} + \delta_{\sigma\rho}\boldsymbol{\lambda}_{\gamma\delta}\cdot\boldsymbol{\lambda}_{\alpha\beta}\right) \\ + &\frac{16}{9}\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{\sigma\rho} - \frac{1}{2}d^{abc}\boldsymbol{\lambda}^a_{\alpha\beta}\boldsymbol{\lambda}^b_{\gamma\delta}\boldsymbol{\lambda}^c_{\sigma\rho} + \frac{1}{2}if^{abc}\boldsymbol{\lambda}^a_{\alpha\beta}\boldsymbol{\lambda}^b_{\gamma\delta}\boldsymbol{\lambda}^c_{\sigma\rho} \end{aligned} \tag{20a}$$

$$\begin{aligned} &\delta_{\alpha\rho}\boldsymbol{\lambda}_{\gamma\beta}\cdot\boldsymbol{\lambda}_{\sigma\delta} + \delta_{\gamma\beta}\boldsymbol{\lambda}_{\alpha\rho}\cdot\boldsymbol{\lambda}_{\sigma\delta} + \delta_{\sigma\beta}\boldsymbol{\lambda}_{\gamma\rho}\cdot\boldsymbol{\lambda}_{\alpha\delta} \\ = &\frac{2}{3}\left(\delta_{\alpha\beta}\boldsymbol{\lambda}_{\gamma\delta}\cdot\boldsymbol{\lambda}_{\sigma\rho} + \delta_{\gamma\delta}\boldsymbol{\lambda}_{\alpha\beta}\cdot\boldsymbol{\lambda}_{\sigma\rho} + \delta_{\sigma\rho}\boldsymbol{\lambda}_{\gamma\delta}\cdot\boldsymbol{\lambda}_{\alpha\beta}\right) \\ + &\frac{16}{9}\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{\sigma\rho} - \frac{1}{2}d^{abc}\boldsymbol{\lambda}^a_{\alpha\beta}\boldsymbol{\lambda}^b_{\gamma\delta}\boldsymbol{\lambda}^c_{\sigma\rho} - \frac{1}{2}if^{abc}\boldsymbol{\lambda}^a_{\alpha\beta}\boldsymbol{\lambda}^b_{\gamma\delta}\boldsymbol{\lambda}^c_{\sigma\rho} \end{aligned} \tag{20b}$$

$$\begin{aligned} d^{abc}\boldsymbol{\lambda}^a_{\alpha\delta}\boldsymbol{\lambda}^b_{\gamma\rho}\boldsymbol{\lambda}^c_{\sigma\beta} = &-\frac{5}{27}\left(\delta_{\alpha\beta}\boldsymbol{\lambda}_{\gamma\delta}\cdot\boldsymbol{\lambda}_{\sigma\rho} + \delta_{\gamma\delta}\boldsymbol{\lambda}_{\alpha\beta}\cdot\boldsymbol{\lambda}_{\sigma\rho} + \delta_{\sigma\rho}\boldsymbol{\lambda}_{\gamma\delta}\cdot\boldsymbol{\lambda}_{\alpha\beta}\right) \\ &+ \frac{80}{81}\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{\sigma\rho} + \frac{13}{18}d^{abc}\boldsymbol{\lambda}^a_{\alpha\beta}\boldsymbol{\lambda}^b_{\gamma\delta}\boldsymbol{\lambda}^c_{\sigma\rho} - \frac{5}{18}if^{abc}\boldsymbol{\lambda}^a_{\alpha\beta}\boldsymbol{\lambda}^b_{\gamma\delta}\boldsymbol{\lambda}^c_{\sigma\rho} \end{aligned} \tag{21a}$$

$$\begin{aligned} d^{abc}\boldsymbol{\lambda}^a_{\alpha\rho}\boldsymbol{\lambda}^b_{\gamma\beta}\boldsymbol{\lambda}^c_{\sigma\delta} = &-\frac{5}{27}\left(\delta_{\alpha\beta}\boldsymbol{\lambda}_{\gamma\delta}\cdot\boldsymbol{\lambda}_{\sigma\rho} + \delta_{\gamma\delta}\boldsymbol{\lambda}_{\alpha\beta}\cdot\boldsymbol{\lambda}_{\sigma\rho} + \delta_{\sigma\rho}\boldsymbol{\lambda}_{\gamma\delta}\cdot\boldsymbol{\lambda}_{\alpha\beta}\right) \\ &+ \frac{80}{81}\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{\sigma\rho} + \frac{13}{18}d^{abc}\boldsymbol{\lambda}^a_{\alpha\beta}\boldsymbol{\lambda}^b_{\gamma\delta}\boldsymbol{\lambda}^c_{\sigma\rho} + \frac{5}{18}if^{abc}\boldsymbol{\lambda}^a_{\alpha\beta}\boldsymbol{\lambda}^b_{\gamma\delta}\boldsymbol{\lambda}^c_{\sigma\rho} \end{aligned} \tag{21b}$$

$$
\begin{aligned}
if^{abc}\boldsymbol{\lambda}^a_{\alpha\delta}\boldsymbol{\lambda}^b_{\gamma\rho}\boldsymbol{\lambda}^c_{\sigma\beta} &= \frac{1}{3}\left(\delta_{\alpha\beta}\boldsymbol{\lambda}_{\gamma\delta}\cdot\boldsymbol{\lambda}_{\sigma\rho} + \delta_{\gamma\delta}\boldsymbol{\lambda}_{\alpha\beta}\cdot\boldsymbol{\lambda}_{\sigma\rho} + \delta_{\sigma\rho}\boldsymbol{\lambda}_{\gamma\delta}\cdot\boldsymbol{\lambda}_{\alpha\beta}\right) \\
&- \frac{16}{9}\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{\sigma\rho} + \frac{1}{2}d^{abc}\boldsymbol{\lambda}^a_{\alpha\beta}\boldsymbol{\lambda}^b_{\gamma\delta}\boldsymbol{\lambda}^c_{\sigma\rho} - \frac{1}{2}if^{abc}\boldsymbol{\lambda}^a_{\alpha\beta}\boldsymbol{\lambda}^b_{\gamma\delta}\boldsymbol{\lambda}^c_{\sigma\rho} \qquad \text{(22a)} \\
if^{abc}\boldsymbol{\lambda}^a_{\alpha\rho}\boldsymbol{\lambda}^b_{\gamma\beta}\boldsymbol{\lambda}^c_{\sigma\delta} &= -\frac{1}{3}\left(\delta_{\alpha\beta}\boldsymbol{\lambda}_{\gamma\delta}\cdot\boldsymbol{\lambda}_{\sigma\rho} + \delta_{\gamma\delta}\boldsymbol{\lambda}_{\alpha\beta}\cdot\boldsymbol{\lambda}_{\sigma\rho} + \delta_{\sigma\rho}\boldsymbol{\lambda}_{\gamma\delta}\cdot\boldsymbol{\lambda}_{\alpha\beta}\right) \\
&+ \frac{16}{9}\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{\sigma\rho} - \frac{1}{2}d^{abc}\boldsymbol{\lambda}^a_{\alpha\beta}\boldsymbol{\lambda}^b_{\gamma\delta}\boldsymbol{\lambda}^c_{\sigma\rho} - \frac{1}{2}if^{abc}\boldsymbol{\lambda}^a_{\alpha\beta}\boldsymbol{\lambda}^b_{\gamma\delta}\boldsymbol{\lambda}^c_{\sigma\rho} \qquad \text{(22b)}
\end{aligned}
$$

III. COMMENTS

Two of the three operators $\sum_{i<j}^{3}\boldsymbol{\lambda}_i\cdot\boldsymbol{\lambda}_j, d^{abc}\boldsymbol{\lambda}^a_1\boldsymbol{\lambda}^b_2\boldsymbol{\lambda}^c_3, if^{abc}\boldsymbol{\lambda}^a_1\boldsymbol{\lambda}^b_2\boldsymbol{\lambda}^c_3$ are SU(3) invariants, i.e. they can be expressed in terms of the two Casimir operators of SU(3) as follows

$$
\sum_{i<j}^{3}\boldsymbol{\lambda}_i\cdot\boldsymbol{\lambda}_j = 2C^{(1)} - 4 \qquad \text{(23a)}
$$

$$
d^{abc}\boldsymbol{\lambda}^a_1\boldsymbol{\lambda}^b_2\boldsymbol{\lambda}^c_3 = \frac{4}{3}\left[C^{(2)} - \frac{5}{2}C^{(1)} + \frac{20}{3}\right]; \qquad \text{(23b)}
$$

where the two Casimir operators of SU(3) are defined as $C^{(1)} = \mathbf{F}^2, C^{(2)} = d^{abc}\mathbf{F}^a\mathbf{F}^b\mathbf{F}^c$ and $\mathbf{F}^a$ are the SU(3) algebra generators. The third operator, $if^{abc}\boldsymbol{\lambda}^a_1\boldsymbol{\lambda}^b_2\boldsymbol{\lambda}^c_3$, is a peculiar object: it is an SU(3) invariant, because it commutes with the SU(3) generators $\mathbf{F}^a = \frac{1}{2}\left(\boldsymbol{\lambda}^a_1 + \boldsymbol{\lambda}^a_2 + \boldsymbol{\lambda}^a_3\right)$ in the special case when these generators are formed from three Gell-Mann matrices, but it is also an off-diagonal operator [it annihilates the two three-quark SU(3) eigenstates with definite exchange symmetry properties, i.e. the **1** and **10**, and turns one **8** state into another] that cannot be expressed in terms of Casimir operators. This result does not violate the Casimir-v.d. Waerden theorem relating the rank of the group to the number of independent invariant operators, as it is representation dependent. This example, however, points out the existence of such invariants which is not widely known.

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